The Parameterized Complexity of Positional Games*

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Abstract : We study the parameterized complexity of several positional games. Our main result is that SHORT GENERALIZED HEX is W[1]-complete parameterized by the number of moves. This solves an open problem from Downey and Fellows' influential list of open problems from 1999. Previously, the problem was thought of as a natural candidate for AW[*]-completeness.

Our main tool is a new fragment of first-order logic where universally quantified variables only occur in inequalities. We show that model-checking on arbitrary relational structures for a formula in this fragment is W[1]-complete when parameterized by formula size.

We also consider a general framework where a positional game is represented as a hypergraph and two players alternately pick vertices. In a Maker-Maker game, the first player to have picked all the vertices of some hyperedge wins the game. In a Maker-Breaker game, the first player wins if she picks all the vertices of some hyperedge, and the second player wins otherwise. In an Enforcer-Avoider game, the first player wins if the second player picks all the vertices of some hyperedge, and the second player wins otherwise.

SHORT MAKER-MAKER is AW[*]-complete, whereas SHORT MAKER-BREAKER is W[1]-complete and SHORT ENFORCER-AVOIDER co-W[1]-complete parameterized by the number of moves. This suggests a rough parameterized complexity categorization into positional games that are complete for the first level of the W-hierarchy when the winning configurations only depend on which vertices one player has been able to pick, but AW[*]-completeness when the winning condition depends on which vertices both players have picked. However, some positional games where the board and the winning configurations are highly structured are fixed-parameter tractable. We give another example of such a game, SHORT k-CONNECT, which is fixed-parameter tractable when parameterized by the number of moves.

Mots-clés : Hex, Maker-Maker games, Maker-Breaker games, Enforcer-Avoider games, parameterized complexity theory

1 Introduction

In a *positional game* [12], two players alternately claim unoccupied elements of the board of the game. The goal of a player is to claim a set of elements that form a winning set, and/or to prevent the other player from doing so.

TIC-TAC-TOE, its competitive variant played on a 15×15 board, GOMOKU, as well as HEX are the most well-known positional games. When the size of the board is not fixed, the decision problem, whether the first player has a winning strategy from a given position in the game is PSPACE-complete for many such

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games. The first result was established for GENERALIZED HEX, a variant played on an arbitrary graph [7]. Reisch soon followed up with results for GOMOKU [14] and HEX played on a board [15]. More recently, PSPACE-completeness was obtained for HAVANNAH [3] and several variants of CONNECT(m, n, k, p, q) [13], a framework that encompasses TIC-TAC-TOE and GOMOKU.

In a Maker-Maker game, also known as strong positional game, the winner is the first player to claim all the elements of some winning set. In a Maker-Breaker game, also known as weak positional game, the first player, Maker, wins by claiming all the elements of a winning set, and the second player, Breaker, wins by preventing Maker from doing so. In an Enforcer-Avoider game, the first player, Enforcer, wins if the second player claims all the vertices of a winning set, and the second player, Avoider, wins otherwise.

In this paper, we consider the corresponding short games, of deciding whether the first player has a winning strategy in ℓ moves from a given position in the game, and parameterize them by ℓ. The parameterized complexity of short games is known for games such as chess [18], generalized geography [1, 2], and pursuit-evasion games [19]. For HEX, played on a hexagonal grid, the short game is FPT and for GENERALIZED HEX, played on an arbitrary graph, the short game is W[1]-hard and in AW[*].

When winning sets are given as arbitrary hyperedges in a hypergraph, we refer to the three game variants as MAKER-MAKER, MAKER-BREAKER, and ENFORCER-AVOIDER, respectively. MAKER-BREAKER was first shown PSPACE-complete by [16] under the name G_{pos}(POS DNF). A simpler proof was later given by [4] who also showed PSPACE-completeness of MAKER-MAKER. To the best of our knowledge, the classical complexity of ENFORCER-AVOIDER has not been established yet.

In this paper we will show that the short game for GENERALIZED HEX is W[1]-complete, solving an open problem stated numerous times [3, 6, 5, 9, 17], we establish that the short game for a generalization of Tic-Tac-Toe is FPT, and we determine the parameterized complexity of the short games for MAKER-MAKER, MAKER-BREAKER, and ENFORCER-AVOIDER. One of our main tools is a new fragment of first-order logic where universally-quantified variables only occur in inequalities and no other relations. After giving some necessary definitions in the next section, we will state our results precisely, and discuss them. The rest of the paper is devoted to the proofs of our results, with some parts deferred to the appendix, due to space constraints.

2 Preliminaries

Finite structures. A vocabulary τ is a finite set of relation symbols, each having an associated arity. A finite structure $\mathcal{A}$ over τ consists of a finite set A, called the universe, and for each R in τ a relation over A of corresponding arity. An (undirected) graph is a finite structure $G = (V, E)$, where E is a symmetric binary relation. A hypergraph is a finite structure $G = (V \cup E, IN)$, where $IN \subseteq V \times E$ is the incidence relation between vertices and edges. Sometimes it is more convenient to denote a hypergraph instead by a tuple $G = (V, E)$ where E is a set of subsets of V.

First-order logic. We assume a countably infinite set of variables. Atomic formulas over vocabulary τ are of the form $x_1 = x_2$ or $R(x_1, \dots, x_k)$ where $R \in \tau$ and $x_1, \dots, x_k$ are variables. The class FO of all first-order formulas over τ consists of formulas that are constructed from atomic formulas over τ using standard Boolean connectives $\neg, \wedge, \vee$ as well as quantifiers $\exists, \forall$ followed by a variable. Let φ be a first-order formula. The size of (a reasonable encoding of) φ is denoted by $|\varphi|$. The variables of φ that are not in the scope of a quantifier are called free variables. We denote by $\varphi(\mathcal{A})$ the set of all assignments of elements of A to the free variables of φ such that φ is satisfied. We call $\mathcal{A}$ a model of φ if $\varphi(\mathcal{A})$ is not empty. The class Σ_1 contains all first-order formulas of the form $\exists x_1, \dots, \exists x_k \varphi$ where φ is a quantifier free first-order formula.

Parameterized complexity. The class FPT contains all parameterized problems that can be decided by an FPT-algorithm. An FPT-algorithm is an algorithm with running time $f(k) \cdot n^{\mathcal{O}(1)}$, where $f(\cdot)$ is an arbitrary computable function that only depends on the parameter k and n is the size of the problem instance. An FPT-*reduction* of a parameterized problem Π to a parameterized problem Π' is an FPT-algorithm that transforms an instance (I, k) of Π to an instance (I', k') of Π' such that: (i) (I, k) is a yes-instance of Π if and only if (I', k') is a yes-instance of Π', and (ii) $k' = g(k)$, where $g(\cdot)$ is an arbitrary computable function that only depends on k. Hardness and completeness with respect to parameterized complexity classes is defined analogously to the concepts from classical complexity theory, using FPT-reductions. The following parameterized classes will be needed in this paper: FPT $\subseteq$ W[1] $\subseteq$ AW[*]. Many parameterized complexity classes can be defined via a version of the following model checking problem.

MC(Φ)
Instance: Finite structure $\mathcal{A}$ and formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

In particular, the problem $\mathrm{MC}(\Sigma_1)$ is W[1]-complete and the problem MC(FO) is AW[*]-complete (see for example [8]).

Positional games. Positional games are played by two players on a hypergraph $G = (V, E)$. The vertex set V indicates the set of available positions, while the each hyperedge $e \in E$ denotes a winning configuration. The two players alternatively claim unclaimed vertices of V until either all elements are claimed or one player wins. The notion of winning depends on the game type. In a *Maker-Maker game*, the first player to claim all vertices of any hyperedge $e \in E$ wins. In a *Maker-Breaker game*, the first player (*Maker*) wins if she claims all vertices of any hyperedge $e \in E$. If the game ends and player 1 has not won, then the second player (*Breaker*) wins. In an *Enforcer-Avoider game*, the first player (*Enforcer*) wins if the second player (*Avoider*) claims all vertices of any hyperedge $e \in E$. If the game ends and player 1 has not won, then the second player wins. A positional game is called an l-move game, if the game ends either after a player wins or both players played l moves. A winning strategy for player 1 is a move for player 1 such that for all moves of player 2 there exists a move of player 1... such that player 1 wins.

3 Results

We view a *positional game* as a hypergraph $G = (V, E)$. Two players alternately claim an unclaimed vertex from V, and the winning sets are the hyperedges E. Depending on the game, the hyperedges are either implicitly or explicitly defined. A *position* in a positional game is an allocation of vertices to the players, who have already claimed these vertices. The *empty position* is the position where no vertex is allocated to a player.

In a *Maker-Maker* game, the winner is the first player to claim all the vertices of some hyperedge. In a *Maker-Breaker* game, the first player, Maker, wins by claiming all the vertices of a hyperedge, and the second player, Breaker, wins by preventing Maker from doing so. In an *Enforcer-Avoider* game, the first player, Enforcer, wins if the second player claims all the vertices of a winning set, and the second player, Avoider, wins otherwise.

The first game we consider is a Maker-Maker game that generalizes well-known games TIC-TAC-TOE, CONNECT6, and GOMOKU (also known as FIVE IN A ROW). In CONNECT(m, n, k, p, q), the vertices are cells of an $m \times n$ grid, each set of k aligned cells (horizontally, vertically, or diagonally) is a winning set, the first move by player 1 is to claim q vertices, and then the players alternate claim p unclaimed vertices at each turn. TIC-TAC-TOE corresponds to CONNECT$(3, 3, 3, 1, 1)$, CONNECT6 to CONNECT$(19, 19, 6, 2, 1)$, and GOMOKU to CONNECT$(19, 19, 5, 1, 1)$. Variations with different board sizes are also common. In the SHORT k-CONNECT problem, the input is the set of $m \cdot n$ vertices, an assignment of some of these vertices to the two players, the integer p, and the parameter ℓ. The winning sets are implicitly defined. The question is whether player 1 has a winning strategy from this position in at most ℓ moves. We omit q from the problem definition of SHORT k-CONNECT since we are modeling games that advanced already past the initial moves. Our first result (proved in Section 4) is that SHORT k-CONNECT is fixed-parameter tractable for parameter ℓ. (In all our results, the parameter is the number of moves, ℓ.)

Theorem 1 SHORT k-CONNECT *is* FPT.

The main reason for this tractability is the rather special structure of the winning sets. It helps reducing the problem to model checking for first-order logic on locally bounded treewidth structures, which is FPT [10].

A similar strategy was recently used to show that SHORT HEX is FPT [3]. The HEX game is played on a parallelogram board paved by hexagons, each player owns two opposite sides of the parallelogram. Players alternately claim an unclaimed cell, and the first player to connect their sides with a path of connected hexagons wins the game. Note that we may view HEX as a Maker-Breaker game: if the second player manages to disconnect the first players sides, he has created a path connecting his sides. [3] also considered a well-known generalization to arbitrary graphs. The GENERALIZED HEX game is played on a graph with two specified vertices s and t. The two players alternately claim an unclaimed vertex of the graph, and player 1 wins if she can connect s and t by vertices claimed by her, and player 2 wins if he can prevent player 1 from doing so. The SHORT GENERALIZED HEX problem has as input a graph G, two vertices s and t in G, an allocation of some of the vertices to the players, and an integer ℓ. The parameter is ℓ, and the question is whether player 1 has a winning strategy to connect s and t in ℓ moves.

The SHORT GENERALIZED HEX problem is known to be in AW[*] and was conjectured to be AW[*]-complete [3, 6, 5, 9, 17]. In fact, AW[*] is thought of as the natural home for most short games [6], playing a similar role in parameterized complexity as PSPACE in classical complexity for games with polynomial length. However, [3] only managed to show that SHORT GENERALIZED HEX is W[1]-hard, leaving a complexity gap between W[1] and AW[*]. Our next result is to show that SHORT GENERALIZED HEX is in W[1]. Thus, SHORT GENERALIZED HEX is in fact W[1]-complete.

Theorem 2 SHORT GENERALIZED HEX *is* W[1]*-complete.*

Our main tool is a new fragment of first-order logic for which model-checking on arbitrary relational structures is W[1]-complete parameterized by the length of the formula. This fragment, which we call $\forall^{\neq}$-FO, is the fragment of first-order logic where universally-quantified variables appear only in inequalities.

Theorem 3 MC($\forall^{\neq}$-FO) *is* W[1]*-complete.*

This result is proved by reducing a formula in $\forall^{\neq}$-FO to a formula in Σ_1. The $\forall^{\neq}$-FO logic makes it convenient to express short games where we can express that player 1 can reach a certain configuration without being blocked by player 2, no matter what configurations player 2 reaches. This is indeed the case for GENERALIZED HEX, where we are merely interested in knowing if player 1 can connect s and t without being blocked by player 2.

More generally, this is the case for SHORT MAKER-BREAKER, where the input is a hypergraph $G = (V, E)$, a position, and an integer ℓ, and the question is whether player 1 has a winning strategy to claim all the vertices of some hyperedge in ℓ moves.

Theorem 4 SHORT MAKER-BREAKER *is* W[1]*-complete.*

The fact that SHORT MAKER-BREAKER is PSPACE-complete and W[1]-complete (and *not* AW[*]-complete) may challenge the intuition one has on alternation. Looking at the classical complexity (PSPACE-completeness), it seems that both players have comparable expressivity and impact over the game. As the game length is polynomially bounded, if the outcome could be determined by only guessing a sequence of moves from one player, then the problem would lie in NP. Now from the parameterized complexity standpoint, SHORT MAKER-BREAKER is equivalent under FPT reductions to guessing the k vertices of a clique (as in the seminal W[1]-complete k-CLIQUE problem); no alternation there. Those considerations may explain why it was difficult to believe that GENERALIZED HEX is *not* AW[*]-complete as conjectured repeatedly [17, 5, 6].

This is also in contrast to SHORT MAKER-MAKER, where the input is a hypergraph $G = (V, E)$, a position, and an integer ℓ, and the question is whether player 1 has a strategy to be the first player claiming all the vertices of some hyperedge in ℓ moves.

Theorem 5 SHORT MAKER-MAKER *is* AW[*]*-complete.*

For the remaining type of positional games, the SHORT ENFORCER-AVOIDER problem has as input a hypergraph $G = (V, E)$, a position, and an integer ℓ, and the question is whether player 1 has a strategy to claim ℓ vertices that forces player 2 to complete a hyperedge. Again, player 1 can only block some moves of player 2, and the winning condition for player 1 can be expressed in $\forall^{\neq}$-FO.

Theorem 6 SHORT ENFORCER-AVOIDER *is* co-W[1]*-complete.*

Our results suggest that a structured board may suggest that a positional game is FPT, but otherwise, the complexity depends on how the winning condition for player 1 can be expressed. If it only depends on what positions player 1 has reached, our results suggest that the problem is W[1]-complete, but when the winning condition for player 1 also depends on the position player 2 has reached, the game is probably AW[*]-complete.

4 SHORT k-CONNECT is FPT

Graph G represents an $m \times n$ board in the following sense. Every board cell is represented by a vertex. Horizontal, vertical and diagonal neighbouring cells are connected via an edge. Vertex sets V_1 and V_2 represent the vertices already occupied by Player 1 and Player 2. While integer p, the number of stones to be placed during a move, is part of the input, we restrict it to values below constant k as games with $p \geq k$ are trivial.

> SHORT k-CONNECT
> *Instance:* A graph $G = (V, E)$ representing an $m \times n$ board, occupied vertices $V_1, V_2 \subseteq V$, and integer p and l.
> *Parameter:* l.
> *Problem:* Decide whether Player 1 has a winning strategy with at most l moves.

Theorem 1 SHORT k-CONNECT *is* FPT.

Proof. We reduce SHORT k-CONNECT to first-order model checking MC(FO) on a bounded local treewidth structure. Using a result by Frick and Grohe [10], it follows that SHORT k-CONNECT is FPT. Let (G, V_1, V_2, p, l) be an instance of SHORT k-CONNECT, where $G = (V, E)$. We construct instance $(\mathcal{A}, \varphi)$ of MC(FO) as follows. Let $EDGE$ be a binary relation symbol and let $V1$ and $V2$ be unary relation symbols. Then $\mathcal{A}$ is the $\{EDGE, V1, V2\}$-structure $(V, EDGE^{\mathcal{A}}, V1^{\mathcal{A}}, V2^{\mathcal{A}})$ with $EDGE^{\mathcal{A}} := E$, $V1^{\mathcal{A}} := V_1$, and $V2^{\mathcal{A}} := V_2$. FO-formula φ is defined as $\varphi \equiv \exists x_1^1 \exists x_1^2 \ldots \exists x_1^p \forall y_1^1 \ldots \forall y_1^p \exists x_2^1 \ldots \exists x_2^p \forall y_2^1 \ldots \exists x_l^p \exists u_1 \exists u_2 \ldots \exists u_k \forall v_1 \forall v_2 \ldots \forall v$

$$\psi \equiv \bigvee_{i=0}^{l} \Big[legalP1_i(x_1^1, \ldots, x_1^p, y_1^1, \ldots, x_l^p) \wedge \Big(\neg legalP2_i(x_1^1, \ldots, x_1^p, y_1^1, \ldots, x_l^p) \vee$$
$$\Big(configP1_i(x_1^1, \ldots, x_l^p, u_1, \ldots, u_k) \wedge \bigwedge_{j=1}^{k-2} aligned(u_j, u_{j+1}, u_{j+2}) \wedge$$
$$\Big(\neg configP2_i(y_1^1, \ldots, y_l^p, v_1, \ldots, v_k) \vee \neg \bigwedge_{j=1}^{k-2} aligned(v_j, v_{j+1}, v_{j+2}) \Big) \Big) \Big) \Big]$$

$$\begin{aligned}
path(u,v,w) &\equiv EDGE(u,v) \wedge EDGE(v,w), \\
hor_vert(u,v,w) &\equiv \exists x \exists y\, path(u,v,w) \wedge path(u,x,w) \wedge path(u,y,w) \wedge path(x,v,y) \wedge \\
&\qquad \forall z \Big[\big(z \neq v \wedge z \neq x \wedge z \neq y \big) \to \neg path(u,z,w) \Big], \\
diag(u,v,w) &\equiv path(u,v,w) \wedge \forall x \Big[x \neq v \implies \neg path(u,x,w) \Big], \\
aligned(u,v,w) &\equiv hor_vert(u,v,w) \vee diag(u,v,w).
\end{aligned}$$

Variables x_i^j represent the jth stone in Player 1's ith move and variables y_i^j represent the jth stone in Player 2's ith move. The sequences $u_1 \ldots u_k$ and $v_1 \ldots v_k$ represent possible winning configurations for Player 1 and Player 2. The overall structure of ψ is the following. The first disjunction ranging from $i = 0$ to $i = l$ represents the number of moves Player 1 needs to win the game. We then ensure that the x variables represent legal moves by Player 1. Further, either variables y do not represent legal moves by Player 2, or Player 1 achieved a winning configuration. For the latter, we assure that variables u represent aligned vertices occupied by Player 1. Finally, we check that Player 2 did not achieve a winning configuration before, that is vertices v do not represent aligned vertices occupied by Player 2.

Formula $path(u, v, w)$ expresses that there is a path of length 2 between vertices u and w via v. Formula $hor\_vert(u, v, w)$ expresses that vertices u, v, and w are aligned horizontally or vertically in this order. A case analysis shows that u, v and w are horizontally or vertically aligned if and only if there are exactly three nodes at distance 1 of u and w, and that v is in the middle of the other two. Formula $diag(u, v, w)$ expresses that vertices u, v, and w are diagonally aligned in this order. This is the case if there exists no other length 2 path between u and w. Formula $aligned(u, v, w)$ expresses that vertices u, v, and w are aligned (in that order). Formula $legalP1_i$ (see Appendix A) ensures that variables x_i^j represent legal moves of Player 1, that is vertices not contained in V_1 or V_2 or previously played vertices. Analogously, $legalP2_i$ ensures that variables y_i^j represent legal moves of Player 2. Formula, $configP1_i$ (see Appendix A) expresses that variables $u_1, \dots, u_k$ form a valid configuration of exactly k vertices out of the set of V_1 or vertices played by Player 1. Analogously, $\mathit{configP2}_i$ states that variables $v_1, \dots, v_k$ form a valid configuration of exactly k vertices out of the set of V_2 or vertices played by Player 2. The size of φ is polynomial in l, k, and p. Since k is a constant and p is bounded by k, we have an FO formula polynomial in our parameter l. Graph G represents a grid with diagonals. Hence, G has maximum degree 8 and therefore bounded local treewidth. It follows from Frick and Grohe [10] that Short Connect is FPT. □

5 MC($\forall^{\neq}$-FO) is W[1]-complete

The class $\forall^{\neq}$-FO contains all first-order formulas of the form $Q_1x_1Q_2x_2Q_3x_3 \dots Q_kx_k\varphi$, with $Q_i \in \{\forall, \exists\}$ and φ being a quantifier free first-order formula such that every $\forall$-quantified variable x_i only occurs in inequalities, that is in relations of the form $x_i \neq x_j$ for some variable x_j.

Theorem 3 MC($\forall^{\neq}$-FO) *is* W[1]*-complete.*

Proof. Hardness: Every Σ_1 formula is contained in the class $\forall^{\neq}$-FO. Hence, W[1]-hardness follows from W[1]-completeness of MC(Σ_1).

Membership: By reduction to MC(Σ_1). Let $(\mathcal{A}, \varphi)$ be an instance of MC($\forall^{\neq}$-FO). If φ contains only existential quantifiers then $(\mathcal{A}, \varphi)$ is already an instance of MC(Σ_1). Hence, let $\varphi = Q_1x_1Q_2x_2 \dots Q_{i-1}x_{i-1}\forall x_i \exists x_{i+1} \exists x_{i+2} \dots \exists x_k \psi$ with $Q_j \in \{\forall, \exists\}$ for $1 \leq j < i$, ψ is in negation normal form and $|\varphi| = l$. That is, x_i is the rightmost of the universal quantified variables. In order to reduce $(\mathcal{A}, \varphi)$ to an instance of MC(Σ_1), we need a way to remove all universal quantifications. We will show how to eliminate the universal quantification of x_i. This technique can then be used to iteratively eliminate all the universal quantifiers. Let $\varphi_1(x_1, \dots, x_{i-1})$ be the subformula $\varphi_1(x_1, \dots, x_{i-1}) \equiv \forall x_i \exists x_{i+1} \dots \exists x_k \psi$. We will show that we can replace $\varphi_1(x_1, \dots, x_{i-1})$ by

$$\varphi_2(x_1, \dots, x_{i-1}) \equiv \exists y_i \exists y_{i+1} \dots \exists y_k \Big(\psi[y_i/x_i, y_{i+1}/x_{i+1}, \dots, y_k/x_k] \wedge \tag{1}$$

$$\bigwedge_{j=1}^{i-1} \exists y^j_{i+1} \exists y^j_{i+2} \dots \exists y^j_k \psi[x_j/x_i, y^j_{i+1}/x_{i+1}, y^j_{i+2}/x_{i+2}, \dots, y^j_k/x_k] \wedge \tag{2}$$

$$\bigwedge_{j=i+1}^{k} \exists y^j_{i+1} \exists y^j_{i+2} \dots \exists y^j_k \psi[y_j/x_i, y^j_{i+1}/x_{i+1}, y^j_{i+2}/x_{i+2}, \dots, y^j_k/x_k] \Big). \tag{3}$$

This reduction is an FPT-reduction, since the size of formula φ_2 is a function of the size of formula φ_1. Let $c_1, \dots, c_{i-1}$ be arbitrary but fixed elements of the universe A of $\mathcal{A}$. We will show that $\varphi_1(x_1, \dots, x_{i-1}) \equiv \varphi_2(x_1, \dots, x_2)$ by proving (a) $\varphi_1(c_1, \dots, c_{i-1}) \to \varphi_2(c_1, \dots, c_{i-1})$ and (b) $\varphi_2(c_1, \dots, c_{i-1}) \to \varphi_1(c_1, \dots, c_{i-1})$. For (a) assume that $\varphi_1(c_1, \dots, c_{i-1})$ is true. This means, $\varphi_1[c_i/x_i]$ is true for all $c_i \in A$, that is for all $c_i \in A$ there exists an assignment to $x_{i+1}, \dots, x_k$ such that ψ is true. Part (1) of $\varphi_2(c_1, \dots, c_{i-1})$ asks for some $c_i \in A$ such that there exists an assignment to $x_{i+1}, \dots, x_k$ such that ψ is true. Part (2) asks for the existence of an assignment to $x_{i+1}, \dots, x_k$ such that ψ is true for each of the cases where x_i is one of the elements $c_1, \dots, c_{i-1}$. Part (3) asks for the existence of an assignment to $x_{i+1}, \dots, x_k$ such that ψ is true for each of the cases where x_i is one of the elements that are assigned to $x_{i+1}, \dots, x_k$ in the model of Part (1). All these are special cases of the universal quantification over x_i, hence $\varphi_2(c_1, \dots, c_{i-1})$ is true.

For direction (b) assume towards a contradiction that $\varphi_1(c_1, \dots, c_{i-1})$ is false and that $\varphi_2(c_1, \dots, c_{i-1})$ is true. Since φ_1 is false, there exists $c_i \in A$ such that $\varphi_1[c_i/x_i]$ is false. We perform a case distinction on the value c_i. First let $c_i = c_j$ for $1 \leq j < i$. Then let $c_{i+1}, \dots, c_k$ be the assignments to variables $y^j_{i+1}, \dots, y^j_k$ in the model of φ_2. The jth conjunct of Part (3) of φ_2 states that ψ holds for $x_i = x_j$ using the assignment $c_{i+1}, \dots, c_k$. Hence, assigning $c_{i+1}, \dots, c_k$ to variables $x_{i+1}, \dots, x_k$ in φ_1 is a model for $\varphi_1[c_i/x_i]$, which contradicts our assumption. As the next case, let $c_{i+1}, \dots, c_k$ be the assignment to variables $y_{i+1}, \dots, y_k$ in the model of φ_2 and let $c_i = c_j$ for $i < j \leq k$. Let $c'_{i+1}, \dots, c'_k$ be the assignments to variables $y^j_{i+1}, \dots, y^j_k$ in the model of φ_2. The jth conjunct of Part (2) of φ_2 states that ψ holds for $x_i = x_j = c_j$ using the assignment $c'_{i+1}, \dots, c'_k$. Hence, assigning $c'_{i+1}, \dots, c'_k$ to variables $x_{i+1}, \dots, x_k$ in φ_1 is a model for $\varphi_1[c_i/x_i]$, which contradicts our assumption. For the last case, let c_i be one of the remaining values. Let $l_1, \dots, l_m$ be all the literals in ψ that contain x_i. All of them are inequalities of the form $x_i \neq x_j$ for $j \neq i$. Let c'_i be the assignment to y_i in the model of φ_2. Let $l'_1, \ldots, l'_m$ be the literals in $\psi[y_i/x_i, y_{i+1}/x_{i+1}, \ldots, y_k/x_k]$ in Part (1) of φ_2 that correspond to $l_1, \ldots, l_m$. We have no knowledge about the truth value of these literals l'_j with $1 \leq j \leq m$, but all of the literals l_j in ψ evaluate to true when assigning $c_{i+1}, \ldots, c_k$ to the variables $x_{i+1}, \ldots, x_k$. Since ψ is in negation normal form and the literals $l_1, \ldots, l_m$ never occur in unnegated form, that is as equalities, changing the truth value of these literal from false to true will never result in changing the truth value of the whole formula from true to false. But since c'_i together with $c_{i+1}, \ldots, c_k$ is a model of Part (1) of φ_2, it holds that for all values of c_i that we consider in this case, that $\varphi_1[c_i/x_i]$ is true, which contradicts our assumption. This completes the case distinction and we have $\varphi_1(x_1, \ldots, x_{i-1}) \equiv \varphi_2(x_1, \ldots, x_2)$. □

6 SHORT GENERALIZED HEX is W[1]-complete

SHORT GENERALIZED HEX	
Instance:	Graph $G = (V, E)$, vertices $s, t \in V$, vertex sets $V_1, V_2 \subseteq V$ with $V_1 \cap V_2 = \emptyset$, and integer l.
Parameter:	l.
Problem:	Decide whether Player 1 has a winning strategy with at most l moves in the generalized Hex game (G, s, t, V_1, V_2).

A generalized Hex game (G, s, t, V_1, V_2) is a positional game (V', E'), where the positions V' and the winning configurations E' are defined as follows. Set V' contains all vertices of G, that is $V' = V$. Set E' contains a set of vertices $\{v_1, \ldots, v_k\}$ if and only if $\{v_1, \ldots, v_k\} \cup \{s, t\}$ form an $s - t$ path in G. Additionally, vertices in V_1 and V_2 are already claimed by player 1 and player 2, respectively. Since the set of winning configurations of SHORT GENERALIZED HEX is only defined implicitly, the input size of SHORT GENERALIZED HEX can be exponential smaller than the number of winning configurations.

Theorem 2 SHORT GENERALIZED HEX *is* W[1]*-complete.*

Proof. Hardness is already known [3]. For membership, we reduce SHORT GENERALIZED HEX to MC($\forall^{\neq}$-FO). Let (G, s, t, V_1, V_2, l) be an instance of SHORT GENERALIZED HEX, where $G = (V, E)$. Claimed vertices V_1 and V_2 can be preprocessed: (i) every $v \in V_1$ and its incident edges are removed from G and the neighbourhood of v is turned into a clique; (ii) every $v \in V_2$ and its incident edges are removed from G. Hence, w.l.o.g. we assume that $V_1 = V_2 = \emptyset$. We construct an instance $(\mathcal{A}, \varphi)$ of MC($\forall^{\neq}$-FO) as follows. Let $EDGE$ be a binary relation symbol and let S and T be unary relation symbols. Then $\mathcal{A}$ is the $\{EDGE, S, T\}$-structure $(V, EDGE^{\mathcal{A}}, S^{\mathcal{A}}, T^{\mathcal{A}})$ with $EDGE^{\mathcal{A}} := E$, $S^{\mathcal{A}} := \{s\}$, and $T^{\mathcal{A}} := \{t\}$. The $\forall^{\neq}$-FO-formula φ is defined as $\varphi = \exists s \exists t \exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \forall y_{l-1} \exists x_l \exists z_1 \exists z_2 \ldots \exists z_l \psi$, with

$$\psi \equiv S(s) \wedge T(t) \wedge \Big(EDGE(s,t) \vee \bigvee_{i=1} \bigvee_{j=1} \Big(EDGE(s, z_1) \wedge EDGE(z_j, t) \wedge path_{i,j}(x_1, \ldots, x_i, z_1, \ldots, z_j) \wedge diff_i(x_1, y_1, \ldots, y_{i-1}, x_i) \Big) \Big),$$

$$path_{i,j}(x_1, \ldots, x_i, z_1, \ldots, z_j) \equiv \bigwedge_{h=1}^{j-1} EDGE(z_h, z_{h+1}) \wedge \bigwedge_{h=1}^{j} \bigvee_{k=1}^{i} z_h = x_k,$$

$$diff_i(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i) \equiv \bigwedge_{1 \leq j < k \leq i} x_j \neq x_k \wedge \bigwedge_{1 \leq j < k \leq i} y_j \neq x_k.$$

The intuition of φ is the following. The variables x_i, y_i, and z_i represent the moves of Short, the moves of Cut, and the ordered (s,t)-path induced by Short's moves, respectively. The variables s and t represent the vertices of the same name. Formula φ expresses that there is either a direct edge between s and t or a s-t path of length j was played. The main disjunctions ($\bigvee$) ensure that we consider wins that take up to l moves, and build s-t path of length up to l. Subformula $path_{i,j}$ will be true if and only if the z variables form a path using only values of the selected values for the x variables. Subformula $diff_i$ ensures that all x variables are pairwise distinct and they are distinct from all y variables with smaller index.

We have $|\varphi| = \mathcal{O}(l^4)$, so this is indeed an FPT-reduction and W[1]-membership follows. □

7 SHORT MAKER-BREAKER is W[1]-complete

SHORT MAKER-BREAKER
Instance: Hypergraph $G = (V, E)$, vertex sets $V_1, V_2 \subseteq V$ with $V_1 \cap V_2 = \emptyset$, and integer l.
Parameter: l.
Problem: Decide whether Player 1 has a winning strategy with at most l if vertices V_1 and V_2 are already claimed by Player 1 and Player 2, respectively.

Theorem 4 SHORT MAKER-BREAKER *is* W[1]*-complete.*

Proof. For membership, we reduce SHORT MAKER-BREAKER to MC($\forall^{\neq}$-FO). Let (G, V_1, V_2, l) be an instance of SHORT MAKER-BREAKER, where $G = (V, E)$ is a hypergraph. Claimed vertices V_1 and V_2 can be preprocessed: (i) every $v \in V_1$ is removed from V and every hyperedge $e \in E$; (ii) every $v \in V_2$ is removed from V and every hyperedge $e \in E$ with $v \in e$ is removed from E. Hence, w.l.o.g. we assume that $V_1 = V_2 = \emptyset$. We construct an instance $(\mathcal{A}, \varphi)$ of MC($\forall^{\neq}$-FO) as follows. Let IN and SIZE be binary relation symbols. Then $\mathcal{A}$ is the $\{\mathit{IN}, \mathit{SIZE}\}$-structure $(V \cup E \cup \{1, \ldots, |V|\}, \mathit{IN}^{\mathcal{A}}, \mathit{SIZE}^{\mathcal{A}})$ with $\mathit{IN}^{\mathcal{A}} := \{(x, e) \mid x \in V, e \in E, x \in e\}$ and $\mathit{SIZE}^{\mathcal{A}} := \{(e, i) \mid e \in E, |e| = i\}$. Hence, the universe of $\mathcal{A}$ consists of the vertices of G, an element for each hyperedge, and an element for some bounded number of integers. The $\forall^{\neq}$-FO-formula φ is defined as $\varphi \equiv \exists x_1 \forall y_1 \ldots \forall y_{l-1} \exists x_l \exists e \exists z_1 \exists z_2 \ldots \exists z_l \psi$, with

$$\psi \equiv \bigvee_{1 \le j \le i \le l} \Big(\mathit{diff}_i(x_1, y_1, \ldots, x_i) \wedge \mathit{SIZE}(e, j) \wedge \bigwedge_{k=1}^{j} \bigvee_{h=1}^{i} z_k = x_h \wedge \bigwedge_{1 \le k < h \le j} z_k \neq z_h \wedge \bigwedge_{k=1}^{j} \mathit{IN}(z_k, e) \Big).$$

The subformula $\mathit{diff}_i(x_1, y_1, \ldots, x_i)$ refers to the subformula with same name used in the proof of Theorem 2. That is, it ensures that all x variables are pairwise distinct and that they are distinct from all y variables with smaller index. The intuition of φ is the following. The variables x_i and y_i represent the moves of Maker and the moves of Breaker, respectively. The variables z_i represent the vertices forming the winning configuration of Maker and e represents the hyperedge of this winning configuration. The first disjunction ensures that we consider wins that take up to l moves. The second disjunction ensures that we consider winning configurations that consist of up to i vertices. After checking that e has the correct size ($\mathit{SIZE}(e, j)$), we encode that the values of the z variables are contained in the hyperedge represented by e and that these variables are pairwise disjoint and selected among the moves of Maker (the x variables).

We have $|\varphi| = \mathcal{O}(l^4)$, so this is indeed an FPT-reduction and W[1]-membership follows.

For hardness, we reduce k-MULTICOLORED CLIQUE to SHORT MAKER-BREAKER. The reduction is essentially the same as the reduction used for showing W[1]-hardness of GENERALIZED HEX [3]. The crucial observation is that the construction of [3] contains only a polynomial number of possible $s - t$ paths. Hence, we can encode every such $s - t$-path as a unique hyperedge denoting a winning configuration in SHORT MAKER-BREAKER. □

8 SHORT MAKER-MAKER is AW[*]-complete

SHORT MAKER-MAKER
Instance: Hypergraph $G = (V, E)$, vertex sets $V_1, V_2 \subseteq V$ with $V_1 \cap V_2 = \emptyset$, and integer l.
Parameter: l.
Problem: Decide whether Player 1 has a winning strategy with at most l if vertices V_1 and V_2 are already claimed by Player 1 and Player 2.

Theorem 5 SHORT MAKER-MAKER *is* AW[*]*-complete.*

Proof. For membership, we reduce SHORT MAKER-MAKER to MC(FO). Let (G, V_1, V_2, l) be an instance of SHORT MAKER-MAKER, where $G = (V, E)$ is a hypergraph. We construct an instance $(\mathcal{A}, \varphi)$ of MC(FO) as follows. Let $\mathit{V1}$, $\mathit{V2}$, and EDGE be unary relation symbols. Let IN be a binary relation symbol. Then $\mathcal{A}$ is the $\{\mathit{V1}, \mathit{V2}, \mathit{EDGE}, \mathit{IN}\}$-structure $(V \cup E, \mathit{V1}^{\mathcal{A}}, \mathit{V2}^{\mathcal{A}}, \mathit{EDGE}^{\mathcal{A}}, \mathit{IN}^{\mathcal{A}})$ with $\mathit{V1}^{\mathcal{A}} := \{x \mid x \in V_1\}$, $\mathit{V2}^{\mathcal{A}} := \{x \mid x \in V_2\}$, $\mathit{EDGE}^{\mathcal{A}} := \{e \mid e \in E\}$, and $\mathit{IN}^{\mathcal{A}} := \{(x, e) \mid x \in V, e \in E, x \in e\}$. Hence, the universe of $\mathcal{A}$ consists of the vertices and the hyperedges of G. The FO-formula φ is defined as $\varphi \equiv \exists x_1 \forall y_1 \ldots \forall y_{l-1} \exists x_l \psi$, with

$$\psi \equiv \bigvee_{i=0}^{l} legalP1_i(x_1, y_1, \ldots, x_l) \wedge \Big(\neg legalP2_{i-1}(x_1, y_1, \ldots, x_l) \vee$$
$$\big(winP1_i(x_1, y_1, \ldots, x_l) \wedge \neg winP2_{i-1}(x_1, y_1, \ldots, x_l)\big)\Big).$$
$$winP1_i(x_1, y_1, \ldots, x_l) \equiv \exists e \forall z EDGE(e) \wedge \big(\neg IN(z, e) \vee V1(z) \vee \bigvee_{j=1}^{i} z = x_j\big),$$
$$winP2_i(x_1, y_1, \ldots, x_l) \equiv \exists e \forall z EDGE(e) \wedge \Big(\neg IN(z, e) \vee V2(z) \vee \bigvee_{j=1}^{i} z = y_j\Big).$$

Variable x_j represent Player 1's jth move and variable y_j represent Player 2's jth move. The first disjunction represents the number of moves i that Player 1 needs to win the game. Formula $legalP1_i$ (see Appendix B) ensures that variables $(x_j)_{1\leq j\leq i}$ represent legal moves of Player 1, that is vertices not contained in V_1 or V_2 or previously played vertices. Analogously, $legalP2_i$ ensures that variables $(y_j)_{1\leq j\leq i}$ represent legal moves of Player 2. Formula $winP1_i$ ensures that Player 1 has won within the i first moves, that is, it has completed a hyperedge with V_1 and variables up to x_i. Analogously, $winP2_i$ ensures that Player 2 has won within the i first moves. We have $|\varphi| = \mathcal{O}(l^3)$ and $|\mathcal{A}| = \mathcal{O}(|G|^2)$, so this is indeed an FPT-reduction and AW[*]-membership follows.

For hardness, we reduce from the AW[*]-complete problem SHORT GENERALIZED GEOGRAPHY on bipartite graphs. The reduction is deferred to the appendix. ☐

9 SHORT ENFORCER-AVOIDER is co-W[1]-complete

> SHORT ENFORCER-AVOIDER
> *Instance:* Hypergraph $G = (V, E)$, vertex sets $V_1, V_2 \subseteq V$ with $V_1 \cap V_2 = \emptyset$, and integer l.
> *Parameter:* l.
> *Problem:* Decide whether Player 1 has a winning strategy with at most l moves if vertices V_1 and V_2 are already claimed by Player 1 and Player 2, respectively.

Theorem 6 SHORT ENFORCER-AVOIDER *is* co-W[1]-*complete.*

Proof. We show that the co-problem of SHORT ENFORCER-AVOIDER is W[1]-complete. The co-problem of SHORT ENFORCER-AVOIDER is to decide whether for all strategies of Enforcer, there exists a strategy of Avoider such that during the first l moves, Avoider does not claim a hyperedge. Again, vertices V_1 and V_2 are already claimed by Enforcer and Avoider, respectively. We prove W[1]-hardness by a parameterized reduction from INDEPENDENT SET and W[1]-membership by reduction to MC($\forall^{\neq}$-FO).

In the W[1]-complete INDEPENDENT SET problem [5], the input is a graph $G = (V, E)$ and an integer parameter k, and the question is whether G has an independent set of size k, i.e., a set of k pairwise non-adjacent vertices. We construct a positional game $G' = (V', E')$ by replacing each vertex of G by a clique of size $k+1$. The vertex set V' has vertices $v(1), \ldots, v(k+1)$ for each vertex $v \in V$, and hyperedges $E' = \{\{v(i), v(j)\} : v \in V \text{ and } 1 \leq i < j \leq k+1\} \cup \{\{u(i), v(j)\} : uv \in E \text{ and } 1 \leq i, j \leq k+1\}$. We claim that G has an independent set of size k if and only if Avoider does not claim a hyperedge in the first k moves in the positional game G' starting from the empty position, that is $V_1 = V_2 = \emptyset$. For the forward direction, suppose $I = \{v_1, \ldots, v_k\}$ is an independent set of G of size k. Then, a winning strategy for Avoider is to claim an unclaimed vertex from $\{v_i(1), \ldots, v_i(k+1)\}$ at round $i \in \{1, \ldots, k\}$. We note that Enforcer cannot claim all the vertices from $\{v_i(1), \ldots, v_i(k+1)\}$, since there are not enough moves to do so, and Avoider does not complete a hyperedge with this strategy. On the other hand, suppose Avoider has a winning strategy in k moves. For an arbitrary play by Enforcer, let $\{v_1(i_1), \ldots, v_k(i_k)\}$ denote the vertices claimed by Player 1. Then, $v_i \neq v_j$ and $v_i v_j \notin E$ for any $1 \leq i < j \leq k$, since Player 1 would otherwise claim all the vertices of a hyperedge. Therefore, $\{v_1, \ldots, v_k\}$ is an independent set of G of size k.

For membership, we reduce to MC($\forall^{\neq}$-FO). Let (G, V_1, V_2, l) be an instance of the co-problem of SHORT ENFORCER-AVOIDER where $G = (V, E)$ is a hypergraph. First we do some preprocessing. We remove all vertices from G that are contained in V_2, that is the vertices already claimed by Avoider. If this results in an empty hyperedge, the instance is a no instance. Otherwise, we remove all hyperedges that contain a vertex in V_1, that is the vertices already claimed by Enforcer, since Avoider will never lose via these edges anymore. Finally, we remove all vertices from G that are contained in V_1. Let $G = (V, E)$ now refer to the outcome of this preprocessing. By construction all vertices of G are unoccupied and some vertices might not be contained in any hyperedge. If G contains less than $2l$ vertices we can solve the problem via brute force in FPT time. Hence, in what follows we assume that there are at least $2l$ unoccupied vertices in G. We construct an instance $(\mathcal{A}, \varphi)$ of MC($\forall^{\neq}$-FO) as follows. Let $EDGE_i$ be a i-ary relation symbol for $1 \leq i \leq l$. Then $\mathcal{A}$ is the $\{EDGE_1, \ldots, EDGE_l\}$-structure $(V, EDGE_1{}^{\mathcal{A}}, \ldots, EDGE_l{}^{\mathcal{A}})$ with $EDGE_i{}^{\mathcal{A}} := \{(v_1, \ldots, v_l) \mid e \in E, |e| = i, e = \{v_1, \ldots, v_l\}\}$, that is $EDGE_i{}^{\mathcal{A}}$ contains all hyperedges of cardinality i. The $\forall^{\neq}$-FO-formula φ is defined as

$$\varphi \equiv \forall y_1 \exists x_1 \forall y_2 \exists x_2 \ldots \exists x_l \; \mathit{diff}_l(y_1, x_1, \ldots, x_l) \wedge \bigwedge_{1 \leq i \leq l} \; \bigwedge_{\{z_1, \ldots, z_i\} \subseteq \{x_1, \ldots, x_l\}} \neg EDGE_i(z_1, \ldots, z_i),$$

where $\mathit{diff}_i(y_1, x_1, \ldots, x_i) \equiv \bigwedge_{1 \leq j < k \leq i} x_j \neq x_k \wedge \bigwedge_{1 \leq j \leq k \leq i} y_j \neq x_k$.

Subformula $\mathit{diff}_i(y_1, x_1, \ldots, x_i)$ ensures that all x variables are pairwise distinct and they are distinct from all y variables with index less or equal theirs. The intuition of φ is the following. The variables x_i and y_i represent the moves of Avoider and the moves of Enforcer, respectively. Avoider wins if the x variables do not cover a whole hyperedge after l moves. We only have to check hyperedges of size up to l. Hence, for each cardinality $i \leq l$, we check for all subsets $z_1, \ldots, z_l$ of the x variables that they do not form a hyperedge. Formula φ does not pose any restrictions on the y variables, that is we do not force Enforcer to pick unoccupied vertices. We call a move by Enforcer that picks an already occupied vertex cheating. To prove correctness, we need to show that whenever Enforcer has a winning strategy σ_E that involves cheating, Enforcer also has a winning strategy σ'_E without cheating. We construct σ'_E as follows. We follow strategy σ_E while σ_E does not perform a cheating move. If the next move would be a cheating move, we play a random unoccupied vertex instead and keep track of this vertex in a new set V_r. The next time we need to select a move, we construct a board state s by removing all vertices in V_r from the picks of Enforcer and query strategy σ_E on this state s. If the answer is an unoccupied vertex, we perform this move normally. If instead the answer is a previously played vertex (which might be in V_r), we play a random unoccupied vertex instead and add it to V_r. Since σ_E was a winning strategy, so is σ'_E. Hence, formula φ does not need to check if the y variables correspond to unoccupied vertices. The construction can be done by an FPT algorithm since for each hyperedge $e \in E$ of cardinality i, we create $i! \leq l!$ entries in the $EDGE_i$ relation. We have $|\varphi| = \mathcal{O}(l^l)$, so this is indeed an FPT reduction and W[1]-membership follows. □

10 Conclusion

We have seen that the parameterized complexity of short positional games depends crucially on whether both players compete for achieving winning sets, or whether the game can be seen as one player aiming to achieve a winning set and the other player merely blocking the moves of the first player. Naturally, blocking moves correspond to inequalities in first-order logic, and our $\forall^{\neq}$-FO fragment of first-order logic therefore captures that the universal player can only block moves of the existential player. Our W[1]-completeness of MC($\forall^{\neq}$-FO) has been used several times in this paper, but our transformation of $\forall^{\neq}$-FO formulas into Σ_1 formulas may have other uses. As a concrete example related to positional games, [3] established that SHORT HEX is FPT by expressing the problem as a FO formula, and making use of Frick and Grohe's meta-theorem [10], similarly as we did in Section 4. This establishes that the problem is FPT but the running time is non-elementary in l. However, we remark that their FO formula is actually a $\forall^{\neq}$-FO formula of size polynomial in l. Our transformation gives an equivalent Σ_1 formula whose length is single-exponential in l, and the meta-theorem of [11] then gives a running time for solving SHORT HEX that is triply-exponential in l.

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A Subformulas for Theorem 1

$$
\begin{aligned}
\mathit{legalP1}_i(x_1^1,\dots,x_1^p,y_1^1,\dots,x_l^p) \equiv \bigwedge_{j=1}^{i}\bigwedge_{t=1}^{p}\Big[\neg\,\mathit{V1}(x_j^t)\wedge\neg\,\mathit{V2}(x_j^t)\wedge\bigwedge_{r=1}^{j-1}\bigwedge_{q=1}^{t}(x_j^t\neq x_r^q)\wedge \\
\bigwedge_{q=1}^{t-1}(x_j^t\neq x_j^q)\wedge\bigwedge_{r=1}^{j-1}\bigwedge_{q=1}^{t}(x_j^t\neq y_r^q)\Big], \\
\mathit{legalP2}_i(x_1^1,\dots,x_1^p,y_1^1,\dots,x_l^p) \equiv \bigwedge_{j=1}^{i-1}\bigwedge_{t=1}^{p}\Big[\neg\,\mathit{V1}(y_j^t)\wedge\neg\,\mathit{V2}(y_j^t)\wedge\bigwedge_{r=1}^{j-1}\bigwedge_{q=1}^{t}(y_j^t\neq y_r^q)\wedge \\
\bigwedge_{q=1}^{t-1}(y_j^t\neq y_j^q)\wedge\bigwedge_{r=1}^{j-1}\bigwedge_{q=1}^{t}(y_j^t\neq x_r^q)\Big].
\end{aligned}
$$

$$
\begin{aligned}
\mathit{configP1}_i(x_1^1,\dots,x_l^p,u_1,\dots,u_k) \equiv \bigwedge_{j=1}^{k}\left[\left(\mathit{V1}(u_j)\vee\bigvee_{r=1}^{i}\bigvee_{q=1}^{p}u_j=x_r^q\right)\wedge\bigwedge_{r=1}^{j-1}u_j\neq u_r\right], \\
\mathit{configP2}_i(y_1^1,\dots,y_l^p,v_1,\dots,v_k) \equiv \bigwedge_{j=1}^{k}\left[\left(\mathit{V2}(v_j)\vee\bigvee_{r=1}^{i-1}\bigvee_{q=1}^{p}v_j=y_r^q\right)\wedge\bigwedge_{r=1}^{j-1}v_j\neq v_r\right].
\end{aligned}
$$

B Subformulas for Theorem 5

$$legalP1_i(x_1, y_1, \dots, x_l) \equiv \bigwedge_{1\le j\le i} \Big[\neg V1(x_j) \wedge \neg V2(x_j)\Big] \wedge \bigwedge_{1\le j<k\le i} \Big[x_j \neq x_k \wedge y_j \neq x_k\Big],$$

$$legalP2_i(x_1, y_1, \dots, x_l) \equiv \bigwedge_{1\le j\le i} \Big[\neg V1(y_j) \wedge \neg V2(y_j)\Big] \wedge \bigwedge_{1\le j<k\le i} (y_j \neq y_k) \wedge \bigwedge_{1\le j\le k\le i} x_j \neq y_k.$$

C AW[*]-hardness of SHORT MAKER-MAKER

Reduction from the AW[*]-complete problem SHORT GENERALIZED GEOGRAPHY on bipartite graphs. From an instance $B = (X \uplus Y, F, v_0), k$ of SHORT GENERALIZED GEOGRAPHY, with $v_0 \in X$, we build a hypergraph $G = (V, E), l$ of size polynomial in $|B|$ which will be an equivalent SHORT MAKER-MAKER instance.

In our reduction, the hypergraph G mainly involves two distinguished vertices $\exists, \forall \in V$ and gadgets corresponding to vertices and edges of B. In the initial setup, the vertex $\exists$ is assumed to have already been claimed by Player 1 and the vertex $\forall$ to have already been claimed by Player 2. Our construction ensures that all the hyperedges of E contain exactly one vertex in $\{\exists, \forall\}$. We thus partition the hyperedges between the ones that can make Player 1 win and the ones that can make Player 2 win.

Formally, G is defined as indicated in Equations (4) and (5). It uses gadgets detailed in the rest of this section. The parameter is linearly preserved from the input parameter: $l = 9(k+1) + 6$.

$$V = \{\exists, \forall\} \cup \bigcup_{u\in X} V^{\exists}(u) \cup \bigcup_{u\in Y} V^{\forall}(u) \cup V_4^D(\{\exists\}) \cup V_4^D(\{\forall\}) \tag{4}$$

$$E = \{\{\forall, a^{v_0}\}\} \cup \bigcup_{u\in X} E^{\exists}(u) \cup \bigcup_{u\in Y} E^{\forall}(u) \cup D_4^{\exists}(\{\exists\}) \cup D_4^{\forall}(\{\forall\}) \tag{5}$$

$V_1 = \{\exists\}, V_2 = \{\forall\}$

C.1 Terminology

A *useless* 3-threat for Player 1 is a 3-threat that can be defended, and for which after the 3-threat and its defense, Player 1 has not achieved anything. Formally, the threat and its defense are two vertices which, once played, do not appear in any other hyperedges that could make one player or their opponent win. Note that those threats can be disregarded for Player 1 but not for Player 2. Indeed, Player 2 could use a series of useless 3-threats to win by delaying the game.

A *losing* 3-threat for a player is a 3-threat that can be met with a counter-attack winning in a constant number of moves; more precisely in at most 6 moves.

A *living* 3-threat is a non losing 3-threat; if it is for Player 1, it should in addition be non useless.

C.2 Delay gadget

As a building block of the forthcoming existential and universal gadgets, we introduce the following delay gadgets where $? \in \{\exists, \forall\}$. If $? = \exists$ (resp. $? = \forall$), we say that the delay gadget belongs to Player 1 (resp. to Player 2).

$$D_1^?(S) := \{S \cup \{?, x_1^S\}, S \cup \{?, x_2^S\}, S \cup \{?, x_3^S\}\} \tag{6}$$

$$D_2^?(S) := \bigcup_{i,j\in[3]} \{S \cup \{?, x_i^S, y_j^S\}\} = \left\{\begin{matrix} S \cup \{?, x_1^S, y_1^S\}, S \cup \{?, x_2^S, y_1^S\}, S \cup \{?, x_3^S, y_1^S\}, \\ S \cup \{?, x_1^S, y_2^S\}, S \cup \{?, x_2^S, y_2^S\}, S \cup \{?, x_3^S, y_2^S\}, \\ S \cup \{?, x_1^S, y_3^S\}, S \cup \{?, x_2^S, y_3^S\}, S \cup \{?, x_3^S, y_3^S\} \end{matrix}\right\} \tag{7}$$

$$D_4^?(S) := \bigcup_{g,h,i,j\in[3]} \{S \cup \{?, x_g^S, y_h^S, z_i^S, t_j^S\}\} \tag{8}$$

$$V_1^D(S) := \{x_1^S, x_2^S, x_3^S\} \tag{9}$$

$$V_2^D(S) := \bigcup_{i\in[3]} \{x_i^S, y_i^S\} = \{x_1^S, x_2^S, x_3^S, y_1^S, y_2^S, y_3^S\} \tag{10}$$

$$V_4^D(S) := \bigcup_{i\in[3]} \{x_i^S, y_i^S, z_i^S, t_i^S\} \tag{11}$$

The elements x_i^S, y_i^S, z_i^S, and t_i^S (with $i \in [3]$) will only appear in the corresponding delay gadgets. For any set S, we will introduce at most one set among $D_1^\forall(S)$, $D_1^\exists(S)$, $D_2^\forall(S)$, $D_2^\exists(S)$, $D_4^\forall(S)$, and $D_4^\exists(S)$. This implies that existing x_i^S and y_i^S (with $i \in [3]$) are well-defined.

Lemma 1
Let $\delta \in \{1, 2, 4\}$ and $S \subseteq V$ be a set of vertices such that $D_\delta^\exists(S) \subseteq E$ (resp. $D_\delta^\forall(S) \subseteq E$). If all vertices of S have been claimed by Player 1 (resp. Player 2), and if no more than one vertex of $V_\delta^D(S)$ has been claimed by the opponent, then she (resp. he) has an unstoppable δ-threat.

Proof. The two statements have identical proofs by switching Player 1 and Player 2. We therefore only give a proof for a delay gadget $D_\delta^\exists(S)$. Assume that Player 1 has played all the vertices of S. Without loss of generality, assume that the vertex claimed by the opponent, if any, is x_1^S. Recall that we assume that $\exists$ has already been claimed by Player 1 and $\forall$ has been claimed by Player 2.

For $\delta = 1$, Player 1 has at least two 1-threats, playing in x_2^S or x_3^S, and Player 2 cannot block them both. Thus, if Player 2 claims x_i^S (with $i \in [3]$), she claims x_j^S with $j \neq i \in [3]$ and wins.

For $\delta = 2$, Player 1 has several 2-threats. If Player 2 claims x_i^S (resp. y_i^S) for some $i \in [3]$, Player 1 claims x_j^S (resp. y_j^S) for some $j \neq i \in \{2, 3\}$ and obtains an unstoppable 1-threat.

For $\delta = 4$, the reasoning is similar and omitted. □

Corollary 1 *Let $\delta \in \{1, 2, 4\}$ and $S \subseteq V$ be a set of vertices such that $D_\delta^\exists(S) \subseteq E$ (resp. $D_\delta^\forall(S) \subseteq E$). If Player 1 (resp. Player 2) claims all vertices in S and no more than one vertex of $V_\delta^D(S)$ has been claimed by the opponent, then if it is that player's turn, they can force a win in δ moves unless the opponent has a $\delta - 1$-threat. If it is the opponent's turn, then Player 1 (resp. Player 2) can force a win in δ moves unless the opponent has a δ-threat.*

C.3 Existential vertex gadget

For each vertex $u \in X$ in the existential partition of the SHORT GENERALIZED GEOGRAPHY instance, we introduce in G the following hyperedges:

$$\begin{aligned} E^\exists(u) = \quad & D_2^\exists(\{a^u, b^u\}) \cup D_2^\forall(\{b^u, e^u\}) \cup D_2^\forall(\{b^u, g^u\}) \\ & \cup \bigcup_{v\in N(u)} D_1^\exists(\{a^u, c_v^u, d_v^u\}) \\ & \cup D_1^\forall(\{b^u, d_v^u, e^u\}) \cup D_1^\exists(\{c_v^u, e^u, f^u\}) \\ & \cup D_1^\forall(\{d_v^u, f^u, g^u\}) \cup D_1^\exists(\{c_v^u, g^u, h^u\}) \\ & \cup D_2^\forall(\{d_v^u, i_v^u\}) \cup D_2^\exists(\{i_v^u, a^v\}) \end{aligned}$$

In terms of vertices of G introduced by the gadget, each vertex $u \in X$ gives rise to a set $V^\exists(u)$ that contains all the vertices needed by the delay sub-gadgets along with $\{a^u, b^u, e^u, f^u, g^u, h^u\} \cup \bigcup_{v\in N(u)}\{c_v^u, d_v^u, i_v^u\}$.

Lemma 2
Consider the gadget for an existential vertex $u \in X$ such that no element of $V^\exists(u) \setminus \{a^u\}$ has been claimed yet. Assume that a^u has been played by Player 1 and that it is Player 2's turn. If Player 2 has no non-losing 3-threats in the whole board, then for each $v \in N(u)$ such that a^v has not been claimed yet, Player 1 has a strategy $\sigma^\exists(u, v)$ that ensures either that Player 2 plays a^v after no more than 8 moves all of which belonging to $V^\exists(u)$ and that there are no non-losing 3-threats left for Player 2 in the gadget or that Player 1 wins in no more than 14 moves.

Proof. We exhibit the strategy for Player 1 and show that Player 2's answers are forced to prevent Player 1 from winning. By assumption Player 2 has no non-losing 3-threats anywhere else on the board and no vertices claimed in $V^{\exists}(u)$, so unless Player 2 play b^u, Player 1 wins in 6 moves by claiming b^u herself via Corollary 1 applied to $D_2^{\exists}(\{a^u, b^u\})$. Although Player 2 has now sets of 3-threats which involve e^u and g^u, he does not have any 2-threats. Player 1 plays c_v^u which forces Player 2 to claim d_v^u by Corollary 1 applied to $D_1^{\exists}(\{a^u, c_v^u, d_v^u\})$. Player 1 plays e^u which forces Player 2 to claim f^u. Player 1 plays g^u which forces Player 2 to claim h^u. Player 1 plays i_v^u. At this point, 8 moves have been played, Player 2 has no 3-threats left in the gadget, so Player 2 is forced to play a^v lest Player 1 plays a^v and wins in a total of 14 moves by Corollary 1 applied to $D_2^{\exists}(\{i_v^u, a^v\})$.

Since Player 1 has claimed e^u, g^u, and i_v^u, the only local hyperedges remaining for Player 2 are $D_2^{\forall}(\{d_w^u, i_w^u\})$ for $w \neq v$, and none of them feature a 3-threat. □

Lemma 3
Consider the gadget for an existential vertex $u \in X$ such that no element of $V^{\exists}(u) \setminus \{a^u\}$ has been claimed yet. Assume that for any vertex $v \in Y$, a^v has not been claimed by Player 1. Assume that a^u has been played by Player 1 and that it is Player 2's turn. If Player 1 has no living 3-threats elsewhere on the board, then Player 2 has a strategy $\sigma^{\forall}(u)$ that ensures either 1) that after no less than 8 moves, all of which either belong to $V^{\exists}(u)$ or are not in any live existential hyperedge, Player 2 plays a^v for some v and there are no living 3-threats left for Player 1, or it is Player 2's turn and there is no living 3-threat for Player 1; or 2) that Player 2 wins.

Proof. We exhibit a local strategy for Player 2, any move by Player 1 in a non-living 3-threat elsewhere on the board is responded to accordingly. Player 2 plays b^u creating sets of 3-threats in $D_2^{\forall}(\{b^u, e^u\})$ and $D_2^{\forall}(\{b^u, g^u\})$. Playing either of e^u and g^u is losing for Player 1 because Player 2 can play in the other vertex. Therefore, Player 1 needs to play in a 3-threat to avoid losing. Notwithstanding the non-living 3-threats, the only 3-threats for Player 1 can be found in the gadgets $D_1^{\exists}(\{a^u, c_v^u, d_v^u\})$ for $v \in N(u)$.

As long as Player 1 plays in d_w^u for some w, Player 2 replies in the corresponding c_w^u voiding the threat. As soon as Player 1 plays a move other than d_v^u in $D_1^{\exists}(\{a^u, c_v^u, d_v^u\})$ for some v, Player 2 can answer d_v^u, voiding the threat, and play proceeds as follows. Player 1 has no 2-threats and so replying e^u is forced to avoid losing via Corollary 1 applied to $D_1^{\forall}(\{b^u, d_v^u, e^u\})$. Player 2 plays f^u which forces Player 1 to claim g^u. Player 2 plays h^u threatening to play i_v^u. Therefore, Player 1 needs to either play in 3-threats via the gadgets $D_1^{\exists}(\{a^u, c_w^u, d_w^u\})$ for some $w \in N(u)$ such that d_w^u has not been claimed yet, or Player 1 has to play in i_v^u herself. As long as Player 1 plays in d_w^u for some w, Player 2 replies in the corresponding c_w^u voiding the threat.

Eventually, Player 1 has to play in i_v^u. If a^v has already been claimed by Player 2, then Player 2 is left with no 3-threat to defend. Otherwise, Player 2 plays a^v. □

C.4 Universal vertex gadget

For each $u \in Y$, we introduce in G the following hyperedges:

$$
\begin{aligned}
E^{\forall}(u) = \quad & D_2^{\forall}(\{a^u, b^u\}) \cup D_2^{\exists}(\{b^u, g^u\}) \cup D_2^{\exists}(\{b^u, i^u\}) \\
& \cup \bigcup_{v \in N(u)} D_1^{\forall}(\{a^u, c_v^u, d_v^u\}) \\
& \cup D_1^{\exists}(\{b^u, d_v^u, e_v^u\}) \cup D_1^{\forall}(\{c_v^u, e_v^u, f^u\}) \\
& \cup D_1^{\exists}(\{b^u, c_v^u, j_v^u\}) \cup D_1^{\forall}(\{d_v^u, j_v^u, f^u\}) \\
& \cup D_1^{\exists}(\{c_v^u, f^u, g^u\}) \cup D_1^{\exists}(\{d_v^u, f^u, g^u\}) \cup D_1^{\forall}(\{e_v^u, g^u, h^u\}) \\
& \cup D_1^{\exists}(\{f^u, h^u, i^u\}) \cup D_2^{\forall}(\{e_v^u, a^v\})
\end{aligned}
$$

In terms of vertices of G introduced by the gadget, each vertex $u \in Y$ gives rise to a set $V^{\forall}(u)$ that contains all the vertices needed by the delay sub-gadgets along with $\{a^u, b^u, f^u, g^u, h^u, i^u\} \cup \bigcup_{v \in N(u)} \{c_v^u, d_v^u, e_v^u, j_v^u\}$.

We observe that the only shared vertices between the different existential and universal gadgets are a^u for $u \in X \cup Y$. For instance, in the universal gadget, each a^v with $v \in N(u)$ is the "starting vertex" of the existential gadget encoding the vertex $v \in X$.

Lemma 4

Consider the gadget for a universal vertex $u \in Y$ such that no element of $V^{\forall}(u) \setminus \{a^u\}$ has been claimed yet. Assume that for any vertex $v \in X$, a^v has not been claimed by Player 2. Assume that a^u has been played by Player 2 and that it is Player 1's turn. If Player 2 has no non-losing 3-threats elsewhere on the board, then Player 1 has a strategy $\sigma^{\exists}(u)$ that ensures either 1) that after no more than 8 moves, all of which belong to $V^{\forall}(u)$, Player 1 plays a^v for some v and there are no non-losing 3-threats left for Player 2 or it is Player 1's turn and there are no non-losing 3-threats for Player 2; or 2) that Player 1 wins in no more than 14 moves.

Proof. We exhibit a local strategy for Player 1. Player 1 plays b^u creating sets of 3-threats in $D_2^{\exists}(\{b^u, g^u\})$ and $D_2^{\exists}(\{b^u, i^u\})$. Claiming either of g^u and i^u is losing for Player 2 because Player 1 can play in the other vertex. Therefore, Player 2 needs to play in a 3-threat to avoid losing. The only non-losing 3-threats for Player 2 can be found in the gadget $D_1^{\forall}(\{a^u, c_v^u, d_v^u\})$ for $v \in N(u)$.

If Player 2 claims d_v^u, Player 1 plays c_v^u, forcing Player 2 to claim j_v^u. Player 1 plays f^u, forcing Player 2 to claim g^u. At this point, Player 1 can play i^u and win by Corollary 1 applied to $D_2^{\exists}(\{b^u, i^u\})$.

If instead of d_v^u Player 2 starts by claiming c_v^u, then Player 1 plays d_v^u, forcing Player 2 to claim e_v^u. Player 1 plays f^u, forcing Player 2 to claim g^u. Player 1 plays h^u, forcing Player 2 to claim i^u. If a^v has already been claimed by Player 1, then Player 1 is left with no 3-threat to defend. Otherwise, Player 1 plays a^v. ☐

Lemma 5

Consider the gadget for a universal vertex $u \in Y$ such that no element of $V^{\forall}(u) \setminus \{a^u\}$ has been claimed yet. Assume that a^u has been played by Player 2 and that it is Player 1's turn.

If Player 1 has no living 3-threats on the whole board, then for each $v \in N(u)$ such that a^v has not been claimed yet, Player 2 has a strategy $\sigma^{\forall}(u, v)$ that ensures either that Player 1 plays a^v after no less than 8 moves all of which either belong to $V^{\forall}(u)$ or are not in any live existential hyperedge and that there are no living 3-threats left for Player 1 in the gadget; or that Player 2 wins.

Proof. We exhibit the strategy for Player 2 and show that Player 1's answers are forced to prevent Player 2 from winning. By assumption Player 1 has no living 3-threats anywhere else on the board and no vertices claimed in $V^{\forall}(u)$, so unless Player 1 plays b^u, Player 2 wins in 6 moves by claiming b^u himself via Corollary 1 applied to $D_2^{\forall}(\{a^u, b^u\})$. Although Player 1 has now sets of 3-threats which involve e^u and g^u, she does not have any 2-threats. Player 2 plays c_v^u which forces Player 1 to claim d_v^u by Corollary 1 applied to $D_1^{\forall}(\{a^u, c_v^u, d_v^u\})$. Player 2 plays e^u which forces Player 1 to claim f^u. Player 2 plays g^u which forces Player 1 to claim h^u. Player 2 plays i^u. At this point, 8 moves have been played, Player 1 has no 3-threats left in the gadget, so Player 1 is forced to play a^v lest Player 2 plays a^v and wins in a total of 14 moves by Corollary 1 applied to $D_2^{\forall}(\{e_v^u, a^v\})$.

Since Player 2 has claimed g^u and i^u, the only local hyperedges remaining for Player 1 are in $D_2^{\exists}(\{b^u, d_w^u, e_w^u\})$ and $D_2^{\exists}(\{b^u, c_w^u, j_w^u\})$ for $w \neq v$, and none of them is feature a living 3-threat. ☐

C.5 Correctness of the reduction

To show that YES SHORT GENERALIZED GEOGRAPHY instances are mapped to YES SHORT MAKER-MAKER instances and that NO instances are mapped onto NO instances, we prove that any Player 1 winning strategy in SHORT GENERALIZED GEOGRAPHY gives rise to a winning strategy for Player 1 in the corresponding SHORT MAKER-MAKER instance, and conversely for Player 2 winning/delaying strategies.

Assume that Player 1 can ensure a win within k moves in SHORT GENERALIZED GEOGRAPHY with strategy τ, and let us construct a strategy σ ensuring a Player 1 win within l moves in SHORT MAKER-MAKER. After Player 1 starts with move a^{v_0}, we use τ, Lemma 2, and Lemma 4 to create σ such that whenever τ prescribes that the token moves from a vertex $u \in X$ to $v \in Y$, we use $\sigma^{\exists}(u, v)$ to leave the u-gadget and enter the v-gadget. When the SHORT MAKER-MAKER game enters a u-gadget with $u \in Y$, we use $\sigma^{\exists}(u)$ to select moves in the gadget until the u-gadget is left and enters a v-gadget with $v \in X$. If a^v is already claimed by Player 1, then Player 2 has no non-losing threats and Player 1 can enter the $D_4^{\exists}(\{\exists\})$ gadget and win by Corollary 1. Otherwise, we then update the SHORT GENERALIZED GEOGRAPHY game with Player 2 moving the token to v. Eventually, the SHORT GENERALIZED GEOGRAPHY game reaches a vertex $u \in Y$ such that all neighbors have been visited before and the game ends. In the SHORT MAKER-MAKER instance, Player 1 follows $\sigma^{\exists}(u)$ and then wins by entering the $D_4^{\exists}(\{\exists\})$ gadget. If τ guarantees that at most $k' \leq k$ moves are played before Player 1 wins, then σ guarantees that at most $9(k'+1) + 6 \leq l$ moves are played before Player 1 wins.

In the case of a NO SHORT GENERALIZED GEOGRAPHY instance, Player 2 has a strategy τ such that either Player 2 wins, or the game goes for longer than k moves. A corresponding SHORT MAKER-MAKER strategy σ can be derived such that either Player 2 wins in the SHORT MAKER-MAKER game, or the game goes for longer than $9(k+1)+6 = l$ moves. The construction is dual to the one above and relies on Lemma 3 and Lemma 5.